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B.A./B.Sc. THIRD SEMESTER EXAMINATION, DECEMBER 2015
SECOND YEAR [BATCH 2014-17]
MATHEMATICS [Hons]
Paper : III

Date : 15/12/2015
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each group]

Group – A

(*Answer* ***any five*** *questions*)

1. a) Let V and W be two vector spaces over the same field F and V is finite dimensional. If $T: V \to W$ is a linear transformation, then prove that $\dim(\operatorname{Ker} T) + \dim(\operatorname{Im} T) = \dim V$. [5]

   b) Let V = $\mathbb{R}^4$ and W be a subspace of V generated by the vectors (1,0,0,0), (0,1,0,0). Find a basis of the quotient space V/W. Verify that $\dim V/W = \dim V - \dim W$. [3]

   c) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(x, y) = (x + y, x)$. Is T invertible? Support your answer. [2]

2. a) The matrix of a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the ordered basis $\{(-1,1,1), (1,-1,1), (1,1,-1)\}$ of $\mathbb{R}^3$ is $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$. Find the matrix of T relative to the ordered basis $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$. [5]

   b) Determine the conditions for which the system of equations
   $$x + y + z = b$$
   $$2x + y + 3z = b + 1$$
   $$5x + 2y + az = b^2$$
   has (i) only one solution, (ii) no solution, (iii) many solutions. [5]

3. a) Prove that an n×n matrix A over a field F is diagonalizable if and only if A has n linearly independent eigen vectors. [5]

   b) Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix, where $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{bmatrix}$. [5]

4. a) Let V be a finite dimensional vector space over the field F and let $\{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and $\beta_1, \beta_2, \ldots, \beta_n$ be any vectors in W. Then prove that there is precisely one linear transformation $T: V \to W$ such that $T(\alpha_i) = \beta_i$ for i = 1, 2,..., n. [5]

   b) Let V be an n-dimensional vector space over the field F and $B = \{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V. Find the matrix of the linear operator T defined by $T(\alpha_j) = \alpha_{j+1}$, j = 1, ...,n–1 and $T(\alpha_n) = 0$ with respect to the ordered basis B. Also prove that $T^n = 0$ but $T^{n-1} \neq 0$. [5]

5. a) Apply the Gram-Schmidt process to the vectors $\beta_1 = (1,0,1)$, $\beta_2 = (1,0,-1)$, $\beta_3 = (0,3,4)$ to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product. [5]

   b) Let V be an inner product space over a field F $(\mathbb{R}$ or $\mathbb{C})$ and $y, z \in V$. If $(x \mid y) = (x \mid z) \forall x \in V$, then show that y = z. [2]

c) Let $\{\beta_1, \beta_2, ..., \beta_n\}$ be an orthonormal set of vectors in an inner product space V. Then for any $\alpha \in V$, prove that $\|\alpha\|^2 \geq \sum_{i=1}^{n} |c_i|^2$, where $c_i = (\alpha | \beta_i), i = 1, 2, ..., n$. [3]

6. a) In an inner product space V prove that $|(\alpha|\beta)| \leq \|\alpha\| \|\beta\|$ for all $\alpha, \beta \in V$. [4]

b) Let V be a finite-dimensional inner product space and f be a linear functional on V. Then show that there exists a unique vector β in V such that $f(\alpha) = (\alpha|\beta)$ for all $\alpha \in V$. [4]

c) Let V be a n-dimensional inner product space, N be a fixed vector and b be a real number. Then the set $\pi = \{x \in V \mid (x|N) = b\}$ forms a hyperplane in V with normal N. [2]

7. a) Let F be a subfield of the complex numbers and let $T: F^3 \to F^3$ be defined by $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$. Verify that T is a linear transformation. What is the rank of T? [2+3]

b) Reduce the quadratic form $2x^2 + 3y^2 + 6xy + 12yz$ to the normal form and find the rank and signature of the quadratic form. [5]

8. a) V be a n-dimensional inner product space. Let $T: V \to V$ be a linear operator. Then prove that the following are equivalent :

i) T is orthogonal

ii) $\|T(x)\| = \|x\|$ for all $x \in V$.

iii) T takes an orthonormal basis to an orthonormal basis. [5]

b) Let U be a linear operator on an inner product space V. Then show that U is unitary if and only if the adjoint U* of U exists and UU* = U*U = I. [3]

c) Let V be a finite dimensional inner product space over a field F and T be a unitary operator on V. Then show that each eigen value of T is of unit modulus. [2]

Group – B

Answer any four questions : [4×5]

9. A variable plane which is at a constant distance 3p from the origin O cuts the axes in A, B, C. Show that the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$ and that of the tetrahedron OABC is $9(x^{-2} + y^{-2} + z^{-2}) = 16p^{-2}$. [5]

10. Prove that the shortest distance between any two opposite edges of the tetrahedron formed by the planes y + z = 0, z + x = 0, x + y = 0, x + y + z = a is $\frac{2a}{\sqrt{6}}$ and that the three lines of shortest distances intersect at the point x = y = z = –a. [5]

11. A variable sphere passes through the points $(0, 0, \pm c)$ and cuts the straight lines $y = x \tan\alpha$, z = c; $y = -x \tan\alpha$, z = –c in the points P, P′. If $PP' = 2a$, a constant, then show that the centre of the sphere lies on the circle z = 0, $x^2 + y^2 = (a^2 - c^2)\operatorname{cosec}^2 2\alpha$. [5]

12. If the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, then find the equations of the other two. [5]

13. Prove that among all central conicoids, the hyperboloid of one sheet is only the ruled surface. [5]

14. Reduce the equation $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy + 2x + 12y + 10z + 20 = 0$ to the canonical form and determine the type of surface it represents. Find the direction of the principal axes. [5]

Answer <u>any three</u> questions : [3×10]

15. a) Find the components of velocity and acceleration of a moving point referred to a set of rectangular axes revolving with uniform angular velocity ω about the origin in their own plane. [6]

    b) Find the law of force in the case in which a particle describes an arc of a circle which passes through the centre of force. [4]

16. a) A particle is projected vertically upwards with a velocity λV in a medium of resistance K (velocity) per unit mass, V being the terminal velocity. Prove that the greatest height attained by the particle is $\frac{V^2}{g}\left[\lambda - \log(1+\lambda)\right]$ after a time $\frac{V}{g}\log(1+\lambda)$. [6]

    b) A point moves along the arc of a cycloid in such a manner that the direction of motion rotates with constant angular velocity; show that the acceleration of the moving point is constant in magnitude. [4]

17. a) A particle moves with a central acceleration F in a medium of which the resistance is $K(\text{velocity})^2$ per unit mass. Show that the differential equation to the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{F}{h_0^2u^2}e^{2Ks}$, where h_0 is the initial angular momentum about the centre of force. [5]

    b) A particle is projected from an apse at a distance 'c' with a velocity $\sqrt{\frac{2}{3}\mu \cdot c^3}$. If the force to the centre be $\mu(r^5 - c^4r)$, show that the equation to the path is $x^4 + y^4 = c^4$. [5]

18. a) A straight smooth tube revolves with constant angular velocity ω in a horizontal plane about one extremity which is fixed. If at zero time a particle inside it be at a distance 'a' from a fixed end and moving with constant velocity V along the tube, then show that its distance at time t is $a\cosh\omega t + \frac{V}{\omega}\sinh\omega t$. [6]

    b) Prove that for a particle moving down a smooth inclined plane, the sum of K.E and P.E is constant. [4]

19. a) One end of an elastic string whose modulus of elasticity is λ and whose unstretched length is 'a', is fixed to a point on a smooth horizontal table and the other end is tied to a particle of mass m which is lying on the table. The particle is pulled to a distance where the extension of the string is b, and then let go. Show that the time of complete oscillation is $2\left(\pi + \frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}$. [6]

    b) A particle describes a parabola $x^2 = 8y$ under a force which is always perpendicular to the y–axis. Find the law of force and velocity at any point of its orbit. [4]

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